TB spaces

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Abstract

This paper considers the basic TB spaces and what can be done with them.

1 Introduction

The term *space* will always mean a topological space that satisfies the T_0 separation axiom: that is, distinct points have distinct neighborhood filters.

The term *map* will always mean a continuous function between spaces; that is, the inverse image of an open set is an open set.

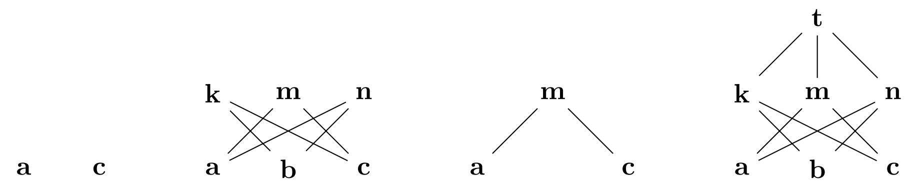
2 Poset results Introduction

A decomposition thm is given for finite partially ordered sets which is an analogue of Birkhoff's thm for lattices (or algebras). Specifically, a spread in a poset is a subset that is not dominated by any single element of the poset, and an upper map is a map that preserves every spread. Our decomposition thm is that every finite post is the upper subdirect product of upper subdirectly irreducible factors.

Characterizations of the irreducible factors are given, and a number of related results are established. Our major result is that two products (finite or infinite) of irreducible factors are isomorphic if and only if their factor sets are identical. This gives rise to a large number of non-isomorphic posets of given large cardinality.

The term poset shall mean a nonempty partially ordered set, of finite cardinality unless otherwise mentioned, and the term map shall mean an order-preserving function between posets.

3 TB factors



$\mathbf{B_2}$ $\mathbf{B_3}$ $\mathbf{T_2}$ $\mathbf{T_3}$

3.1 Spreads

Each subset B of a poset P determines a lower section $\uparrow B = \{a : a \leq b$ for some $b \in B\}$. The subsets of P are pre-ordered by domination, where $B \lhd A$ if $(\uparrow a) \subset (\uparrow B)$ (equivalently $A \subset (\uparrow B)$. A subset that is not dominated by any singleton is called a spread, and it is a minispread if it does not dominate any other spread.

A map is upper if the image of a spread is a spread.

3.2

(a) Every spread dominates a minispread. (This holds only in finite posets). (b) A map is upper if and only if the image of every minispread is a spread. (Only in finite posets). (c) In any finite or infinite product of posets a subset is a spread if and only if at least one of its projections (onto the factors of the product) is a spread.

Prototypes for posets with spreads are found in special subsets of Boolean cubes. Specifically in the Boolean m-cube B_m the atoms are the elements with a single coordinate equal to 1, and the other $m-1$ coordinates equal to 0, and the maximal elements (also called co-atoms) are the elements with a single coordinate of 0 (and the other $m-1$ coordinates equal to 1). We write A_m for the set of atoms, and M_m for the set of maximal elements. A poset is called an m-factor if it is isomorphic to a subset Q of B_m such that $A_m \cup M_m \subset Q \subset B_m \setminus 1$, where 1 is the element each of whose coordinates is 1.

Proposition 1. *An m-factor has a single minispread and it is A_m.*

Proof 1. *For each $a \in P$ let $M(a)$ be that set of maximal elements that are not greater than or equal to a, and set $N(a)$ equal to its complement. Then the subset A is a spread if and only if P is the union of the sets $N(a)$ over all $A \in A$, that is the collection covers P.*

If A is a minispread then for each $a \in A$ the set $A \setminus \{a\}$ is not a spread, so there is a maximal element $m(a)$ that dominates it, and in fact $A \setminus \{a\} \subset N(a)$.

Given any spread $A \subset P$ of cardinal m define a function ϕ from P into the Boolean m-cube B_m, thought of as indexed by the elements of A, using the rules $\phi(t)_a = 0 if x \in N(a)$ and $\phi(t)_a = 1 if x \notin N(a)$. Since A is a spread every *element is in some $N(a)$, so every element in the image of P under ϕ has a 0 in at least one coordinate. That is, $1 \notin S$. Since $\phi(a)_a = 1$ only the element 1 of B_m dominates $\phi[A]$ and so it is a spread in S.*

Now suppose A is a minispread. For each $a \in A$ the set $A \setminus \{a\}$ is not a spread, so there is a maximal element $m(a)$ that dominates it. Such an element has all coordinates except $\phi(a)_m$ equal to 1 and thus a maximal element in the Boolean cube, so S contains the maximal elements of the cube. Since $m(a)$ dominates $A \setminus \{a\}$ then that set is under $N(a)$ which means $\phi(b)_a = 0$ for each $b \in (A \setminus \{a\}$, so $\phi(b)$ is an atom in B_m, and thus S contains the atoms of B_m.

The mapping ϕ just constructed is called the Boolean realization of the spread A. The preceding results are summarized in the following:

Proposition 2. *For any spread in a poset there is a mapping from the poset onto an m-factor whose image is the set of atoms of the m-factor.*

3.3 Maximally ordered posets

Clearly maximal elements and spreads are closely realted, and posets in which the maximal elements play a major role are of great interest. There are three types of posets we will consider: finite posets are always of the first type, but not often of the second or third type.

Definition 1. *A poset is* maximally dominated *if every element is less than or equal to a maximal element.*

Definition 2. *A poset is* maximally separated *if distinct elements are dominated by distinct sets of maximal elements.*

Proposition 3. *If $a \leq b$ then $M(b) \subset M(a)$.*

Definition 3. *A poset is* maximally ordered *if $M(b) \subset M(a)$ implies $a \leq b$.*

Proposition 4. *An upper map with maximally ordered domain is an embedding.*

Proof 2. *Let ϕ be an upper map with maximally ordered domain P.*

Suppose $a \not\leq b$. Then $M(b) \not\subset M(b)$ so there is a maximal element $m \in M(a)$ with $m \notin N(b)$, so that m, a is a spread, and thus the image $\{\phi(m), \phi(a)\}$ is a spread, giving $\phi(a) \not\leq \phi(b)$. Since $\phi(b) \leq \phi(m)$ it follows that $\phi(a) \not\leq \phi(m)$.

3.4 Quotient to maximally ordered

There is a universal mapping from an arbitrary poset to a maximally ordered poset, given by just using the ordering determined by the maximal elements, and this universal mapping is an upper map. Contrast this with EARLIERPROP. The construction is as follows:

Define a relation by $a \leq_\mu b$ if $M(b) \subset M(a)$, and observe readily that this is an equivalence relation that is compatible with the original partial ordering. Then it is readily seen that:

Proposition 5. *(a)the quotient poset of the equivalence relation $\leq_\mu$ is maximally ordered*

(b)the map onto it is an upper map

(c)a poset is maximally ordered if and only if this map is the identity

(d)every map between posets maps also between their quotients.

THAT IS WE HAVE A FUNCTOR.

Proof 3. *a) Clearly distinct maximal elements of P are in distinct equivalence classes, and these classes are precisely the maximal elements of the quotient space. To see that the quotient is maximally ordered not that if $M(a)$ and $M(b)$ are elements of the quotient then their order relation is determined by the order relation of a and b, for any choices of a and b.*

b) It is clear that the quotient map is upper, since if the set A in the original set is not dominated by the element m then the quotient set is not dominated by the maximal element $M(a)$.

c) If P is maximally ordered then the equivalence class $M(a)$ is just the same as the singleton a and the orderings are the same.

d) We need only show that if a and b are equivalent in P then $\psi(a)$ and $\psi(b)$ are equivalent. Now if $\psi(a) \neq \psi(b)$ then there is a maximal element n of Q so that $\psi(a) \leq n$ and $\psi(b) \not\leq n$, and since ψ is a surjection then there is a maximal element $m \in P$ with $\psi(m) = n$. Certainly $b \not\leq m$. Since $\{\psi(a), \psi(m)\}$ is not a spread and ψ is upper then $\{a, m\}$ is not a spread, so $a \leq m$, and thus $a \not\leq_m ub$.

A particular interesting, and we shall see fundamental, type of maximally ordered poset is the m-indecomposable poset.

Definition 4. *A poset is* m-indecomposable *if it has a single minispread of cardinal m that is dominated by every other spread.*

Proposition 6. *A poset is m-indecomposable if and only if it is isomorphic to an m-factor.*

Proof 4. *Any subposet of a Boolean cube that contains all maximal elements and does not contain 1_subm is maximally ordered. If it also contains all atoms then they form a minispread dominated by every other spread, so such a poset is indecomposable.*

If a poset is m-indecomposable then the realization of its unique minispread is an upper map, and thus an embedding, and its image is an m-factor.

3.5 Decomposition

The poset P is the (upper) subdirect product of the set $\{Q_a \; a \in A\}$ of factors if there is a(n upper) map from P into the product that is an embedding, and all projection maps are surjections. The poset is (upper) subdirectly irreducible if in any embedding as an (upper) subdirect product one of the project maps must be an isomorphism of posets.

If we consider only subdirect products then it is well known that any poset is the subdirect product of factors each of which is just the Boolean poset B_1, and that the only subdirectly irreducible posets are the one-point poset and the 2-point chain, which is again the Boolean poset. Clearly an embedding into a product of Boolean factors is upper if and only if the embedded poset has a largest element. Of course such a poset is maximally dominated, and it is maximally separated or maximally determined if and only if it has only a single point.

To obtain an upper embedding into a set of m-factors we must first construct a complement to the equivalence relation $=_\mu$.

Definition 5. *An element of a maximally dominated poset is* free *if it is the largest element of the poset, or if there is another element beneath it that is dominated by the same maximal elements (we say that they free the element above). The set of free elements above a given element will be written as $F(a)$.*

We now define a new ordering relation, which will be compatible with the original relation, and complementary to the ordering determined by maximal elements, by $a \leq_\phi b$ if $F(b) \subset F(a)$. The stated properties are readily established, and we write P_ϕ for the quotient poset of this equivalence relation, and ϕ_ν for the projection of P onto the quotient.

Now we consider the map ϕ of P onto the product poset defined by the two projects ϕ_μ and ϕ_ν.

Proposition 7. *The poset P_ν has a largest element and thus has a representation as a subdirect product of Boolean factors.*

Proof 5. *Note first that if P has a largest element then its equivalence class will be empty and in fact the largest element of P_ν. And if P has no largest element then none of its maximal elements are free (it may not have any), so none of its maximal elements are dominated by a free elements, so the maximal elements are all equivalent to one another, and their class is again the largest element.*

Proposition 8. *The map ϕ is an upper map, and an embedding, giving a representation of P as a subdirect product of a Boolean cube and a maximally determined poset.*

Proof 6. *Since the projection onto P_μ is already known to be upper then the map ϕ is certainly upper. To see that it is an embedding we need only to show that it is an injection.*

Suppose $a \not\leq b$ yet $\phi_m u(a) = \phi_m u(b)$. We have $M(a) = M(b)$ and either a or b must be free, say b, so we have $b \notin F(a)$ and $b \in F(b)$ so $F(a) \neq F(b)$, that is, $\phi_\nu(a) \neq \phi_\nu(b)$.

Proposition 9. *Every maximally ordered poset has an upper subdirect embedding into a product of m-factors.*

Proof 7. *For each minispread in the poset construct its realization into an m-factor. This leads to a map into a product of m-factors, and this map is upper and since its domain is maximally ordered it is an embedding.*

The above results lead directly to

Theorem 1. *Every partially ordered set may be embedded as an upper subdirect product of upper subdirectly irreducible partially ordered sets. The irreducibles are the on-point poset, the Boolean poset, and the m-factors for various integers* m*.*

4 TB-spaces

We are going to use the more descriptive name $TB - space$ for the m-factors, where T reminds us of its top M_m and B reminds us of its bottom A_m. There may also be a 0_m element, consisting of all 0's.

4.1 separating points

We are interested in producing embeddings into products where the products have as much structure as possible. In particular, noting that $\mathbf{T_2}$ is easily seen to be a retract of every proper TB-space, we would like to avoid the two 2-spaces $\mathbb{S}$ and $\mathbf{B_2}$. The latter is easily avoided, it will only occur when there are clopen sets involved, but it is harder to avoid the $\mathbb{S}$ factors.

The following result settles the question.

We first note that a surjection to $\mathbf{T_2}$ is defined by a proper partition into 3 sets $A, B$and V where the first two are a closed disjoint pair, and the third the complement of their union.

Proposition 10. *Distinct points* $\{x\ ney\}$ *can be separated by surjections onto* $\mathbf{T_2}$ *is and only if neither of the two pairs* $\downarrow x, \downarrow y$ *or* $\uparrow x, \downarrow y$ *partitions the space* X*.*

The first case is the standard partition into two clopen sets, and forces the use of $\mathbf{B_2}$*, while the second case partitions into a closed set and its complement and forces the use of* $\mathbb{S}$*.*

Proof. We can assume that $x \notin \downarrow y$. Then if also $y \notin \downarrow y$ we can take the sets $\downarrow x, \downarrow y$ and the complement of their join to define the surjection. and in the other case interchange y with any point not in $\uparrow x$ to define the surjection.

□

It is clear that the second condition cannot hold for a T_1 space, and thus only $\mathbf{B_2}$ factors for the quasicomponents and $\mathbf{T_2}$ factors need to be used.

4.2 products of m-factors/TB-spaces

The following result describes a remarkable property of products of m-factors, the product determines completely the set of factors. Thus the products possess a share of the irreducibility of the factors. We shall later see that subspaces of the product do not share this property.

Theorem 2. *If we have two (finite or infinite) indexed sets of m-factors then the products are isomporphic as posets if and only if there is a bijection between the index sets so that corresponding factors are isomorphic posets.*

The prf breaks into two portions, one applying to disconnected m-factos and the other to all remaining m-factors. The only disconnected m-factor is the discrete poset which we call D_2.

Lemma 1. *The product of an indexed set of factors of D_2 has exactly 2^m order-theoretic components, where m is the cardinality of the index set. Thus if the products are isomorphic then the index sets have the same cardinality, and a bijection between them of course exists.*

To establish notation for the remainder of the prf, which involves maximally ordered sets, suppose we have two indexed sets of factors $\{X_\alpha : \alpha \in \mathcal{A}\}$ and $\{Y_\beta : \beta in \mathcal{B}\}$, and an isomorphism θ between their products X and Y. We denote the projections from the product to the factors for X as π_α.

Choose a fixed minimal element $x \in X$ and set $y = \theta(x)$.

Given any index $\alpha \in \mathcal{A}$ define a mapping $\delta_\alpha : X_\alpha \to X$ by the rule $\pi_\gamma \delta_\alpha(c) = \pi_\gamma(x)$ for $\gamma \in (A), \gamma \neq \alpha$, and $\pi_\alpha \delta_\alpha(c) = c$.