Open Filter Completions

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Abstract

This paper considers a number of classes of spaces and ways of constructing them via open filter.

1 Introduction

1.1 terminology

The term **space** will always mean a topological space that satisfies the $\boldsymbol{T}_0$ separation axiom: that is, distinct points have distinct neighborhood filters.

The symbol $\boldsymbol{O}(X)$ will always stand for the collection of open subsets of $\boldsymbol{X}$, and $\boldsymbol{O}^+(X)$ will be the nonempty members.

The term **map** will always mean a continuous function between spaces; that is, the inverse image of an open set is an open set.

We will occasionally deal with **proper** maps, which are maps for which the inverse image of a compact set is compact, and **closed** maps, which are those for which the image of a closed set is a closed set. Both terms are sometimes used with slightly different definitions, especially when the spaces involved are $\boldsymbol{T}_2$.

The **closure** of a point of a space will be written as $\downarrow \boldsymbol{x}$, and the **saturation** (that is, the intersection of all neighborhoods of) a point will be written as $\uparrow \boldsymbol{x}$. The $\boldsymbol{T}_0$ spaces are precisely those in which we have ($\uparrow x \cap \downarrow x = x$. Equivalently distict points have distinct neighborhoods. We shall from now on assume all spaces to be $\boldsymbol{T}_0$.

1.2 goals

The original goal for a completion theory [?RefWorks:2380] was to seek a unified method of embedding a topological space into another with more specialized properties, both to examine when this could be done and to use the embedding to deduce additional properties or relations to other spaces. The approach of structures followed the very powerful and succesful path laid down by theories of completion of uniform spaces and especially of embedding spaces into compact Hausdorff spaces. A general theory was laid down, following many earlier approaches, in terms of a general structure of collections of open covers and open filters, but the detailed approach was restricted to Hausdorff spaces.

At the time few examples were familiar of what could be done without that restriction. It was known that certain compact $\boldsymbol{T}_0$ spaces arose as the prime ideal spaces of commutative rings, but very little was known about them as general topological spaces and especially about them as possible extensions. It was known that partially ordered sets provided interesting examples of $\boldsymbol{T}_0$ mostly not Hausdorff spaces, but they were not viewed in the context of extensions, except of course for the very succesful theory of the Dedekind-MacNeille completion. The fundamentakl revolution brought about eventually by Dana Scott with injective $\boldsymbol{T}_0$ spaces was just beginning, and the very powerful description of spectral spaces by Melvin Hochster was not yet on the scene.

Following [?RefWorks:2380] a theory of nearness [?RefWorks:1845] was developed, for $\boldsymbol{T}_1$ spaces, but was difficult to apply more broadly both because of lack of examples and because of the lack of powerful methods involving maps of non-regular spaces.

1.2.1 map extensions

The map extension problem is very simple when dealing with a domain that is regular ($\boldsymbol{T}_3$), since all we need is a limit for the image in the codomain for each trace filter. Without this regularity all hell can break loose, and one goal of this paper is to add some understanding to these issues, and suggest some approaches when domains are not regular.

To summarize

- We have a space $\boldsymbol{X}$ embedded via a map $\boldsymbol{\Phi}$ into a space $\boldsymbol{Y}$.
- We have a space $\boldsymbol{Z}$ and map $\boldsymbol{\Psi} : X \to Z$ for which we desire an extension $\boldsymbol{\Phi} : Y \to Z$.
- We have for each point $\boldsymbol{y} \in Y$ the trace filter $\boldsymbol{O}^y$ on $\boldsymbol{X}$ of its neighborhood filter.

Proposition 1. *If Φ exists then the image converges.*

The following result, usually called **extension by regularity**, is in every topology text and automatic to most of us when we deal with extensions.

Proposition 2. *If $\boldsymbol{Z}$ is $\boldsymbol{T}_3$, then when $\boldsymbol{\Phi}^{\#}(O_y)$ converges for each $\boldsymbol{y} \in Y$, assigning the limit as the value defines a continuous extension.*

This result probably goes back to Leopold Vietoris (and perhaps Tietze) in their pioneering work on the ideas of filter, regularity, and compactness developed in 1913-1919. See [?RefWorks:2383] for more detail.

A proof is given here only to illustrate its utter simplicity.

Proof. Defining $\boldsymbol{\Phi}$ as indicated let $\boldsymbol{V}$ be a neighborhood of $\boldsymbol{\Phi} y$ and using regularity let $\boldsymbol{U}$ be open and $\boldsymbol{B}$ closed with $\boldsymbol{\Phi} y \in U \subseteq B \subseteq V$. Then $\boldsymbol{\Phi} U \subseteq B \subseteq V$. □

1.2.2 space extensions

The problem of extensions themselves is equally daunting, and perhaps one reason that most researchers have only dealt woth extensions whose existence can be shown by strong methods (and hypotheses) that are available from more general settings found in categorical or logical argments.

The paper [?RefWorks:2348] shows that without some restriction practically anything can happen:

- Consider for $\boldsymbol{T}_1$ spaces the problem of embedding $\boldsymbol{X}$ into a compact space $\boldsymbol{Y}$ with a given outgrowth $\boldsymbol{Y} \setminus X$.
- If $\boldsymbol{X}$ is not discrete then any nonempty space can be the outgrowth.
- If X is discrete then any nonempty compact space can be the outgrowth.

Thus some restriction is required to have an interesting situation! The present paper attempts to comsider what we have learned in the past decades about nonT_2 extensions, both in terms of examples and potential applications, to see what types of general extension approaches, both to extensions and the maps between them, might be most useful. A general theory of structures which encompasses all the extensions herein discussed has been developed and will be published elsewhere. It includes

- structure spaces [?RefWorks:2380]
- nearness spaces [?RefWorks:1845]
- syntopogeneous spaces [?RefWorks:2337]
- merotopic spaces [?RefWorks:2349]
- binding spaces [?RefWorks:2347]
- and has close relations to most other theories.

1.3 ideas and terminology

- Structures
- Cauchy filters
- Round filters

- It will no longer be the case that there is a single round cauchy filter associated with points of a completion. It is often useful (although not formalized) tp think of a spike associated with the cauchy property representing a relation to the covers (which are called gauges) in the structure, and the point of roundness is to represent waypoints along that spike.

1.4 constraints

- The general topic of topological completion arises in an enormous variety of fields, and very powerful methods have been developed. To name only a few we have the general categorical approach, including especially the theory of monads and related topics. There is of course lattice theory in general, and order-theoretic approaches such as the theory of domains. We will try to avoid deep involvement in these, but often reference them as providing raw material and examples that a topological theory must take into account. It is not expected that the topological approach explored here will provide substantial new material in those fields, but perhaps will give a unifying perspective.
- Many volumes could be (and are) filled with items that relate to our topic. We are going to give only a few basic references. Much of what we develop has been developed elsewhere, sometimes with slightly different definitions, and with different goals. Usually the intent is to elucidate category theory, or domain theory, or an algebraic or order theoretic strucure of some sort, and we are naturally led not to focus on topological ideas that so not fit. Our goal is to try to avoid that shift.
- We have to begin by noting that many works by Bernhard Banaschewski relate from every direction to the current work, and many of them are referenced herein, and many more could well be included.
- Martin Escardo [`?RefWorks:2339`] gives a good intro to the filter monad
- Marcel Erne's The ABC of Order and Topology [`?RefWorks:2346`] is just what it says.
- Escardo summarizes core compact in [`?RefWorks:2342`].

2 strict extension construction and properties

The prototypical $\boldsymbol{T}_0$ space is of course **Sierpinski space**, the two-point space $\{0, 1\}$, symbolized by $\mathbb{S}$, with **0** closed and **1** open.

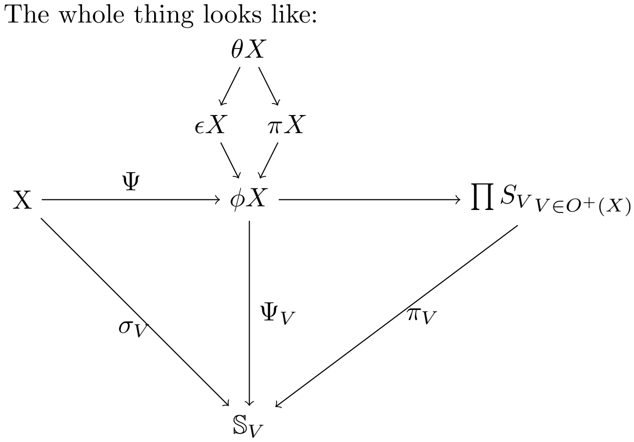
A space $\boldsymbol{X}$ is embedded in its **Sierpinski hull** which is the product of one copy $\mathbb{S}_V$ of Sierpinski space for each nonempty open set $\boldsymbol{V}$ with the standard product topology. We symbolize this space by $\mathbb{S}(X)$ and write $\boldsymbol{\Psi}$ for the embedding map.

The Sierpinski hull turns out to enclose all the important extensions that we will be considering, including:

- the injective extension $\boldsymbol{\phi} X$ [?RefWorks:806],
- the spectral extension $\boldsymbol{\pi} X$ [?RefWorks:2336],
- the essential completion $\boldsymbol{\epsilon} X$ [?RefWorks:2358],
- the sobrification $\theta \boldsymbol{X}$, considered originally by the algebraic topologists.

It also includes other well-known extensions such as the Wallman and Stone-Cech compactifications when they exist. Since it really represents just collections of open subsets, it can also represent covers, filters, and similar items, although we will not pursue this for now.

The whole thing looks like:



Definition 1.

- The *strict filter extension* of a space is constructed with respect to any collection of filters that includes the neighborhood filters. It has as ground set an index set $\boldsymbol{\psi} X$ for the set $\{O^p : p \in \psi X\}$ of those filters.
- Given any $\boldsymbol{V} \in O(X)$ the set $\boldsymbol{V}^*$ is the set of those filters that contain $\boldsymbol{V}$. Noting that $(U \cap V)^* = U^* \cap V^*$ the subbase is closed under meet and thus the collection of all such sets can be taken as a base for open sets to create **the strict topology** on $\boldsymbol{X}$.

  Discussion of this typeof extension is given in [?RefWorks:2330]. [?RefWorks:2315].
- As pointed out in [?RefWorks:2356] the strict extensions of $\boldsymbol{X}$ are exactly the subspaces of $\mathbb{S}X$ that contain $\boldsymbol{X}$, and the largest of these is $\boldsymbol{\phi} X$. As we shall see this extension has the unusual property that it is a retract of $\mathbb{S}X$.

- An *extension* of $\boldsymbol{V}$ is any subset $\cup\{(V_\alpha)^* : \alpha \in A\}$ where $\cup V_\alpha = V$

  The collection of extensions of a particular $\boldsymbol{V}$ is closed under meet and union, and has a largest element $\boldsymbol{V}^*$. It may or may not not have a smallest element.

- An *expansion* of $\boldsymbol{V}$ is any subset $\cup\{(V_\alpha)^* : \alpha \in A\}$ where $\cup V_\alpha$ encloses $\boldsymbol{V}$. The collection of expansions of a particular $\boldsymbol{V}$ is closed under meet and union and forms a filter, which may or may not have a smallest element.

- When the class of filters we are using is preserved under $\sharp$-mapping then the obvious approach takes maps to maps between strict extensions and gives a functor.

- The remarkable properties of these extensions are due in substantial degree to the fact that the specialization order and the inclusion order of the filters are the same: that is

  - $(O^p \subseteq O^q) \equiv (p \leq q)$
  - Furthermore by definition of the strict topology the specialization ordering on $\boldsymbol{X}$, which is just the inclusion ordering of neighborhood filters of points, is the same as the ordering of their filters extended, and the ordering of trace filters and limits agrees.

2.1 retracts of strict extensions

This section is the heart of the paper, and may seem puzzling at first for those thinking in terms of famiiar completions such as the Stone-Cech or Wallman compactifications. After all. retracts of Hausdorff spaces are closed, so a space that is a retract of a completion must have already been complete! And in the case of the Wallman compactification again since a retract (or even an image) of a compact space is compact, and thus already its own Wallman compactifiction, there is nothing to say.

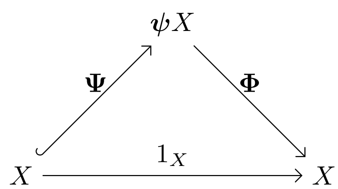
We shall nevertheless consider the situation when $\boldsymbol{X}$ is a retract of some completion $\boldsymbol{\psi}X$ and show that for strict extensions this is equivalent to the single property that *every trace filter has a largest limit.* Thus this single property gives us a characterization of:

- injective spaces when we consider open filters
- spectral spaces when we conider prime filters
- essentially complete spaces when we consider join filters
- sober spaces when we consider union filters

- conditionally complete spaces when we consider saturation filters

Traditional completions are also included here, although not in a very illuminating fashion.

When the completion process is trivial, that is $\boldsymbol{\psi} X = X$ then the retraction is just the identity map, and similarly when the completion process is idempotent, that is $\boldsymbol{\psi}\psi X = \psi X$.



2.2 a largest limit

To say that a point is **the largest limit** of a filter just says that the point is a limit, and is larger in specialization order than any other limit. Thus for $\boldsymbol{T}_1$ spaces in which that order is trivial it says nothing new. For strict extensions in other settings it says quite a lot. We need a couple of lemma to establish this.

The first lemma looks at how an open set gets into a trace filter.

Lemma 1.

- *(i) If $\boldsymbol{V} \in O^p$ then some open subset of the extension contains $\boldsymbol{p}$ and has trace $\boldsymbol{V}$.*
- *(ii) Case (i) certainly occurs if $\boldsymbol{O}^p$ has a limit in $\boldsymbol{V}$.*

Lemma 2. *For every strict extension:*

- *If $\boldsymbol{V}$ does not meet $lmtO^p$ then for every element $\boldsymbol{x} \in V$ there is is an open set $\boldsymbol{V}_x \subseteq V$ whose extension $\boldsymbol{V}_{\boldsymbol{x}}^*$ does not contain $\boldsymbol{p}$, and thus $\cup {V_x}^*$ is an extension that does not contain $\boldsymbol{p}$.*

Theorem 1. *A space is a retract of its strict filter extension with respect to a set of trace filters if and only if every trace filter has a largest limit.*

Remark 1. Each direction requires proof. We shall see this is very much not true for **JF** or **UF**, for example, but it is true for **OF** and **PF**, each when dealing with the appropriate class of spaces.

Proof. We certainly have $\boldsymbol{\Phi}[\mathrm{lmt}(O^p)] \subseteq\downarrow p$ Thus when $\boldsymbol{\Phi}$ is a map we also have $\mathrm{lmt}O^p \subseteq\downarrow \Phi p$.

We also have for $\boldsymbol{V} \in\downarrow p$ that $\boldsymbol{\Phi}^{\leftarrow} V$ is a neighborhood of $\boldsymbol{p}$ so its inverse $\boldsymbol{V} \in O^p$, so $\boldsymbol{\Phi} p$ is a limit of $\boldsymbol{O}^p$.

Conversely the union of all $\boldsymbol{U}* \subseteq \Phi^{\leftarrow} V$ is an extension of $\boldsymbol{V}$, so by the hypthesis and the second lemma it contains $\boldsymbol{p}$.

□

This argument fails in cases where there may not be a largest limit, for example with the Wallman compactification. It works of course in Hausdorff spaces, where limits are unique and so trivially largest, but of course here retracts are closed, so trivial for dense extensions.

Remark 2. It must be pointed out that this retraction gives us extendibility of maps from a space to a strict extension, but this kind of extension has nothing to do with the properties of the map itself, it is just the composition of the original map with the retraction. Nevertheless it can give insight into conditions under which a more interesting extension might exist.

Looked at in another way, the retraction allows us to extend a map to a strict extension just by the properties of the filters considered, and then via the retraction make it an extension into the original codomain.

3 Open Filters

We begin with just naming a variety of types of open filters, and setting forth an ordering between the types. Much of our work will involve considering two cases: one in which two of these varieties are identical, which will characterize many types of spaces, and another in which filters of a particular variety converge, which will characterize other types of spaces.

We next consider limits of open filters in two ways. First we consider the set of limits as a subspace, which will be special types of closed sets, and then weconsider types of open filter and the characteristics of their limit sets.

3.1 open filters

We will just use the term filter, since we do not consider any other type.

- An *open filter* (symbolized **OF**) is a nonempty collection of nonempty open sets which contains the meet of two members, and the join of any open set with a member (equivalently any open set that encloses a member). We will usually be considering a set of open filters which are indexed by some set $\{p\}$ so we write a typical open filter as $\boldsymbol{O}^p$.
- Given a map $\boldsymbol{\Psi}$ the $\# - image$ $\Psi^{\#} O^p$ of a filter $\boldsymbol{O}^p$ is all open sets in the codomain whose inverse image belongs to the filter.

3.2 the dual of a filter

- What we are defining might in some contexts be called an ideal and described more in lattice-theoretical terms, but we are specifically attempting to avoid those more formal settings because they are usually associated with assuming types of completeness that we do not assume. So we will just define the **dual** of an open filter as the collection of closed sets whose complement does not belong to the filter.

  If the filter is $\boldsymbol{O}^p$ then we symbolize its dual as $\hat{\boldsymbol{O}}^p$.

- The dual contains $\boldsymbol{X}$ and does not contain the empty set.

- The dual contains supersets of members.

- If the dual contains a meet of closed sets it contains a member.

- The most important property of the dual is that the intersection of its members is the limit set of the filter. This holds for all open filters independently of their other properties and is often the key to what we need to know about them, in a sense replacing the use of the cluster points for regular spaces.

Proof. It is a trivial argument using the axiom of choice that a filter comverges if and only if it contains a member of every open cover. This same argument shows that a point is a limit if and only if it is in every closed set that is not in the filter. □

3.3 limit sets and filters

- A set is *generic* if it is the closure of a point. Equivalently it is not the union of proper closed subsets.

- A set is *irreducible* if it is not the join of proper closed subsets.

- A set is *essential* if when the entire space is the meet of a finite collection of closed sets then one of them encloses it.

- A *largest limit* of a filter is a generic set which is the set of limits of the filter. This most emphatically does not mean the filter itself is a **NF**, as we see very simply with the eample space $\phi\omega\downarrow$.

- We may as well note here the typical way in which the existence of a largest limit is used. In dealing with coherence and similar issues we might need to show for two $\uparrow -$sets$A$ and $\boldsymbol{B}$ that a particular filter that has a limit in $\boldsymbol{A}$ and has a limit in $\boldsymbol{B}$ has a limit in their intersection. Well, if it has a largest limit it is in each set, and we are done!

We also note that often we do not need a largest limit, but merely that the set of limits is $\uparrow$ −directed Of course in sober spaces the two conditions are the same.

3.4 neighborhood filter

- A **neighborhood filter** (symbolized **NF**) is the collection of all open sets containing a certain point.
- Equivalently it is all open sets that meet a particular generic set.
- It is easily seen that a filter is a **NF** if and only if the intersection

3.5 union filter

- A *union filter* (symbolized **UF**) is an open filter with the additional property that if any union of open sets is a member then one of the sets is a member.
- Equivalently any member contains a limit of the filter (each member meets the limit set).
- Such filters are often called *completely prime*.

3.6 prime filter

- A *prime filter* (symbolized **PF**) is an open filter with the additional property that if a meet of open sets is a member then one of the sets is a member.
- The $\sharp$-image of a **PF** is a **PF**.
- A filter is prime if and only if its dual contains meets of its members. In more common terms, its dual is a prime closed filter (written **cPF**).
- For those who like to use ultrafilters, a prime open filter and its dual are respectively the open members and the closed members of an ultrafilter. There are usually many ultrafilters that give the same pair. We will not develop these connections further in this paper.

3.7 saturation filters

There is one class of filters, and a special subclass, that serve as motivating tools for much of this paper, and the concepts underlying them are historically perhaps the first notions related to a formal theory of completion. Not surprisingly they assume greater importance when we restrict our study to completions of ordered sets.

- A *saturation filter* (symbolized by **SF**) is the collection of all open sets that enclose a nonempty set $\boldsymbol{S}$, which is called a *generating set* for the filter. Clearly the saturation of the set $\boldsymbol{S}$ also generates the same filter.

  A neighborhood filter is the simplest version of a **SF**, motivates the definition, and is the only type available in a $\boldsymbol{T_1}$ space. The filters we deal with in this paper are for the most part not saturation filters.

- A limit of a saturation filter in the saturation topology is precisely any lower bound for the generating set, and in fact this is often a convenient way of viewing such sets topologically.

- A very important situation is when a saturation filter has a largest limit for then the generating point is precisely a least upper bound for the generating set. This notion, of a filter turning out to have a generic limit set, is precisely the way in which we characterize the classes of injective, stably compact, essentially complete, and sober spaces, and also many other classes of spaces such as especially the spectral spaces.

- A very unusual property for this class of filters, that does not hold for any of the other classes we consider, is that every **SF** on a space has a largest limit if and only if this is true for every space in its specialization class. In fact although the specific sets that are members of the filter changes as we move through the spaces in the class, the underlying generating set need not change, and so the set of lower bounds remains the same.

- A nice property satisfied by the limit set of a **SF** is that the set as a whole is not only a closed set but it is an intersection of generic sets. This is, somewhat surprisingly at first glance, not a general property of all closed sets. Of course in a $\boldsymbol{T}_1$ space there is nothing remarkable, it is a single point or empty.

- One might hope that a generating set would have the dual property that it is an intersection of saturations. The obvious generating set, the saturation of $\boldsymbol{S}$ does not have this property, but a larger set, the collection of upper bounds of the limit set, is in fact the largest generating set and has the property.

3.8 Dedekind filters

- The facts relating to the previous remark lead us to define a subclass of **SF** which we call the *Dedekind filters*, which are best viewed as a pair $(L : U))$ in which $\boldsymbol{L}$ is the set of lower bounds of $\boldsymbol{U}$ and $\boldsymbol{U}$ is the set of upper bounds of $\boldsymbol{L}$. Clearly the pair $(\downarrow x :\uparrow x)$ for a point $\boldsymbol{x}$ of a $\boldsymbol{T}_0$ space is the motivating example, and instructive in that it shows that the sets $\boldsymbol{L}$ and $\boldsymbol{U}$ can meet. It is clear they can meet in at most one point, literally by the fact that the space is $\boldsymbol{T}_0$. (In a point-free setting such as locales, which we will not consider here, there is a similar result). For our purposes we also require that neither set by empty.

- The most important property of these filters is the way the ordering works, for we have:

  For **DF** filters $\boldsymbol{O}^p$ and $\boldsymbol{O}^q$ we have $\boldsymbol{O}^p \subseteq O^q$ iff $\boldsymbol{L}_p \subseteq L_q$ iff $\boldsymbol{U}_q \subseteq U_p$. The strict completion with **DF** trace filters is (almost) the Dedekind MacNeille completion, when $\boldsymbol{X}$ is a poset, and it does in fact preserve either a bottom or a top if the space has one already.

- In a $\boldsymbol{T}_1$ space both $\boldsymbol{L}_p$ and $\boldsymbol{U}_p$ can only be singletons, and therefore the same singleton, so the filter is a **NF**. Notice in this case the DM completion has only a top and a bottom.

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3.9 join filter and finite join filter

- A *join filter* (symbolized **JF**) is an open filter that is the join of a collection of neighborhood filters. That is, each member is the intersection of a finite collection of open sets, each of which contains a limit of the filter.

- We call it a *finite join filter* (symbolized $\boldsymbol{f}JF$) if it is the meet of a finite collection.

- The $\# -$ image of a **JF** is a **JF**.

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  Remark 3. It is easy to form a join filter, but for it to be a union filter requires precisely that the intersection of two sets each of which contains a limit of the filter must itself contain a limit, and this is often not the case.

- The definitions of **SF** and **JF** are both generalizations of a property of points, the first of considering all opens that enclose a set, and the second of all opens that meet a set, which of course are the same when the set is a singleton, but not in general. Neither of these should be confused with the idea of a cluster point, which is in the closure of all members and not of much interest when the spaces involved are not regular.

- **Proposition 3.** *A join filter is the union of finite join filters.*

  Proof. Any element of the filter is the meet of open sets that each contains a limit, and thus the filter is the union of elements that belong to an included **fJF**. □

- A **JF** is generated as the meets of opens which meet its limit set.

- Thus a **fJF** is generated by a meet of generics that is also essential.

3.10 zero-set filters and the Stone-Cech compactification

A closely related case, added only for completeness, is the class **GF** of duals of maximal filters of zero sets, discussed beautifully in the classic Gilman-Jerison [?RefWorks:2340]. The correspoonding extension of course is the Stone-Cech compactification.

A filter is a $\boldsymbol{G}F$ filter if and only if for every member $\boldsymbol{U}$ there is a cozero set $\boldsymbol{W}$ such that $\boldsymbol{U} \cup W = X$ and $\boldsymbol{W}$ is not in the filter.

3.11 maximal closed filters and the Wallman compactification

We are generally avoiding the (very) special case of $\boldsymbol{T}_1$ extensions. but will add one special class of filters and the associated extension because it fits in nicely to the general discussion.

A filter is a $\boldsymbol{W}F$ filter if and only if for every member $\boldsymbol{U}$ there is an open set $\boldsymbol{W}$ such that $\boldsymbol{U} \cup W = X$ and $\boldsymbol{W}$ is not in the filter. Equivalently, it is an open filter whose dual ia a maximal closed filter.

It is not the case that the #-image of a **WF** is one as well, but it is the case for the special class of maps that have a closed extension to the Wallman compactification. This class was discussed in Harris [?RefWorks:2334] and [?RefWorks:2333], and these maps form a category on which the Wallman compactification is an epireflection.

Here we also have that $\boldsymbol{X}$ is a retract of $\boldsymbol{\omega}X$ if and only if each of these filters has a generic limit, but of course the whole thing is trivial in $\boldsymbol{T}_1$ spaces.

Proof. TO BE ADDED □

4 specialization class

A basic way of classifying spaces, that is especially useful when we are considering spaces that may not satisfy particular separation axions, is in accordance with their point closures(that is, their generic closed sets). Here we include two spaces in the same class if they have the same generic sets, or equivalently the same **specialization order**, which is a partial order defined by $(\boldsymbol{x} \leq y)$ if $(x \in\downarrow y)$.

Remark 4. The term comes from the fact that the open sets that enclose a point are sometimes regarded as its *properties* and thus a point with less neighborhoods is *less specialized.* This makes sense because the intersection of finitely many properties is reasonably thought of as a property, and the union of any set of properties with a property of a point can still be seen as a property of the point.

The specialization class of any space $\boldsymbol{X}$ contains a largest and a smallest member. All spaces in a class of course have the same ground set.

4.1 the saturation topology

The largest topology has as open sets all unions of saturations and we write this topological space as $\boldsymbol{X}^{\vee}$ and call it **the saturation topology of $\boldsymbol{X}$**. Its unique characteristic is that the open sets are unions of saturations, which are called $\uparrow -\boldsymbol{sets}$ and its closed sets are unions of generics, which are called $\downarrow -\boldsymbol{sets}$.

4.2 the generic topology

The smallest topology has a a subbase for closed sets all the generic sets, or equivalently as closed base all closures of finite sets, and is written as $\boldsymbol{X}^{\wedge}$ and called **the generic topology of $\boldsymbol{X}$**.

4.3 the firm topology

There is a topology usually strictly between these two, often encountered with slight variations in its definitions, and known in its most commonly encountered form as *the Scott topology* We shall call it **the firm topology**, symbolize it by $\boldsymbol{X}^{\nabla}$, and give a general definition that applies to every space.

Define a set to be **firm** if it contains all points that are lower bounds for the set of up-bounded up-directed subsets. Then:

- every generic set is firm
- every join of firm sets is firm
- every intersection of firm sets is firm

- the firm sets form the closed sets of a topology in the same class as X
- and to start us off every set in the generic topology is firm.
- An extremely important property of firmness is that the inverse image of a firm set under an order preserving function is also firm. In our current language this says that the continuity of $\mathbf{\Phi} : X^{\vee} \to Y^{\vee}$ ensures the continuity of $\mathbf{\Phi} : X^{\nabla} \to Y^{\nabla}$. In categorical terminology we have a co-reflection.

  Proof. To see this start with a bounded $\uparrow$-directed $D \subseteq X$ and note that its image in $\boldsymbol{Y}$ is also bounded and $\uparrow$-directed. Then clearly any lower bound for all of the upper bounds of $\boldsymbol{D}$ in $\boldsymbol{X}$ will have an image that is below all upper bounds of its image, and so this holds for the inverse image. We have established that the inverse image of a firm set is firm. $\square$

When the poset $\boldsymbol{X}^{\vee}$ is complete then every $\uparrow$-directed set is bounded and has a sup, and the firm topology is **the Scott topology** introduced in [?RefWorks:2354]. We now have a hierarchy of topologies on the ground set $\boldsymbol{X}$, all having the same point-closures, and can write $\boldsymbol{X}^{\vee} \to X^{\nabla} \to X^{\wedge}$.

There is a weaker situation in which the firm and Scott topologies agree. In a **dcpo** [SEE LATER] every $\uparrow-$directed set has a sup, and that says it converges to a point in the complement of a firm set if and only if it is eventually in that complement, which is the literal definition of an open set in the Scott topology.

$\boldsymbol{X}^{\vee}$ is discrete if and only if $\boldsymbol{X}$ is $\boldsymbol{T}_1$ if and only if $\boldsymbol{X}^{\wedge}$ has finite complement topology (in particular when the set $\boldsymbol{X}$ is finite all these $\boldsymbol{T}_1$ topologies are the same.

Looked at another way the preceding says that very interesting non-$\boldsymbol{T}_1$ topologies are found from non-discrete partially ordered sets as $\boldsymbol{X}^{\vee}$.

4.4 the point of the specialization class

It is very useful to see the spaces of the specialization class as just leaving the two fundamental constants associated with a point, its saturation and its closure, unchanged. We can add open (and thus closed) sets as long as they do not get so small as to reduce these, amd thus move clear to the satiuration topology, amd we can remove open sets (and their complements)as long as we do not lose the generic sets which make up the subbase for the generic topology.

REDO THIS WHOLE SECTION!!!!!!!!!!!!!

4.5 domains

A poset which is sober is called a *dcpo*, or the term *domain* is usually used. There are wide variations in the terms used, so one should always be aware of the particulars.

- Samson Abramsky and Achim Jung in Domain Theory [?RefWorks:2329] gives a solid introduction and many results.

- [?RefWorks:2344] is a very simple introduction.
- For the category of domains we usually consider only maps that preserve the sup of $\uparrow -$directed maps.

An outstanding open question is under what conditions will the firm topology also be sober?

- Ho and Zhou [?RefWorks:2331]
- Johnstone REF and John Isbell [?RefWorks:2367] continues their work to show that even a complete lattice may not be sober in its Scott topology.
- Hui Kou REF shows conditions for a dcpo to be sober.
-

4.6 round filter

This means "Every member is selected". The general plan for structures is to introduce the ideas of "cauchy" filter and "round filter" and treat the "round cauchy" filters as the most prominent objects of interest, both for spaces and for map extension. Cauchy filters are thought of vaguely as the ones that converge, and round filters as the ones that should represent the points of the extension of interest. The trick then is to find the appropriate sructure to use, and the appropriate definition of roundness.

In the case of $\boldsymbol{T}_2$ spaces limits of course are unique, so it might seem that roundness is not important, but that ignores the importancef the extension: roundness tells us where to aim so that the extension behaves as we wish, and the behavior of being $\boldsymbol{T}_2$ is not very automatic. The roundness adopted in [?RefWorks:2380] is one that produces Hausdorff extensions.

In the case of $\boldsymbol{T}_1$ extensions, although roundness is not used specifically, minimality (for open filters, which is identical to maximality for the dual) is used, which is the only possibility since neighborhood filters in $\boldsymbol{T}_1$ spaces are minimal elements in the saturation order.

The main issue with general $\boldsymbol{T}_0$ spaces is that we do not have uniqueness or even extremality to guide us; there very likely will be filters in between that are necessary and no guide to choose them.

A very powerful way to define roundness of a filter is through **selection by set** where we say the member U is selected for O^p if $\boldsymbol{U} \in O^p$ and $U \subseteq S \subseteq V$

where $\boldsymbol{S}$ is:

- compact: to produce local compactness
- closed: to produce regularity

- closed compact: to produce compact-$\boldsymbol{T}_2$ style behavior
- pointed: in relation to injectivity
- a saturation: when dealing with neighborhood filters
- clopen (closed and open): when dealing with zero-dimensional spaces

This type of roundness is very strong, and it can sometimes be achieved by the choice of structure alone, as with nearness where is is just that there is a gauge that selects the member as the only element in the gauge and the filter(clearly this will have problems when the points are not an antichain!).

4.7 selection in strict extensions

There is a form of selection that is appropriate for strict extensions, and all of the selections just listed are appropriate forms for some class of filters and the related extensions, which includes all the familiar examples, often selection by set as just described. $\boldsymbol{U}$ selects V is written $\boldsymbol{U} << V$ and is defined as $\boldsymbol{U}^* \subseteq V^0$, where $\boldsymbol{V}^0$ is defined as the intersection of all extensions of $\boldsymbol{V}$. Equivalently it is the saturation of the set $\boldsymbol{V}$ in the strict filter extension, and is usually not itself an open set.

This set can be defined without mention of the strict extension by that an extension of $\boldsymbol{V}$ is just the union of the $\boldsymbol{V}_\alpha^*$ over any decomposition of $\boldsymbol{V}$, and $\boldsymbol{V}_{\boldsymbol{\alpha}}^*$ is all trace filters that contain $\boldsymbol{V}_\alpha$.

There is a particular meaning for this type of selection that varies with the class of filter involved. Some of these meanings include:

- **OF**: $\boldsymbol{U} \subseteq V$
- **UF**: $\boldsymbol{U}$ has a lower bound in $\boldsymbol{V}$
- **PF**: Any union of open sets that encloses $\boldsymbol{V}$ has a subcollection whose join encloses $\boldsymbol{U}$
- **JF**: $\boldsymbol{U}$ is the meet of sets that each contain a limit of the filter
- **GF**: $\boldsymbol{U} \subseteq Z \subseteq V$ where $\boldsymbol{Z}$ is a zero set
-

John Isbell [?RefWorks:1241] gives a contruction of the zero-dimensional extension with **ZF** as the structure space of uniformly continuous functions into the two-point discrete space.

4.8 conditionally complete spaces and the saturation filter extension

MORE TO COME LATER We define γX to be the strict completion using the saturation filters. Since the saturation order is trivial in a $\boldsymbol{T}_1$ space the only saturationfilters there are **NF**s and so this extension is just $\boldsymbol{X}$. It is quickly seen in general that $\boldsymbol{X}$ is equal to $\boldsymbol{\gamma} X$ if and only if every saturation filter has a largest limit, and that is just equivalent to saying that the space $\boldsymbol{X}$ is conditionally complete, that is, every set bounded below has a greatest lower bound.

There are several unusual situations here.

The first is that we cannot get so-called "universal bounds", that is, a bottom or a top in this manner, although existing ones are prserved.

Note that our definition requires a non-empty generating set, and the existence of any limit at all requires a lower bound for the generating set, so $\boldsymbol{\emptyset}$ never is a generating set, and a bottom can only be obtained with $\boldsymbol{X}$ as the generating set.

The second unusual situation is that the specific space $\boldsymbol{X}$ does not matter, since the limit depends only on the saturation of the generating set, which depends only on the specialization class.

A third interesting difference between the extension $\boldsymbol{\gamma} X$ and the others we consider is that it can be a $\boldsymbol{T}_d$ space, and especially when with a partially ordered set the extension is one as well.

5 sober spaces and the union filter extension

Definition 2. A space is called *sober* if every closed-irreducible set is generic.

We immediately have the characterization:

Proposition 4. *A space is sober if and only if every union filter is a neighborhood filter (*$\boldsymbol{U}F \subseteq NF$*).*

The strict extension formed with the union filters is symbolized as θX, and is a functor from the category Top0 to the category **Sbr** of sober spaces, often called the **sobrification of** $\boldsymbol{X}$.

There are those who confine their attention only to sober spaces, but this of course leaves out the important subcategory of partially ordered sets, in which many important spaces are not sober. The sober spaces are the objects of the category of Locales and in dual form the category of Frames. The product in the categories is quite more complex than the topological product, as is seen in the case of the product of two finite complement spaces, for which the irreducibles of the product space are a set with rather complex structure.

6 injective spaces and the open filter space

Definition 3. A space is called **injective** if it is a retract of any space in which it is densely embedded.

This suggests that a space is injective if and only if it is a retract of its Sierpinski hull. This is of course true, but there is a much stronger result: the largest strict extension, which is the one formed by all open filters, is always a retract of this hull, and thus the spaces that are themselves injective are those that are retract of this extension $\boldsymbol{\phi}X$.

The injective spaces were originally brought to prominence by Dana Scott who emphasized their importance as the topological spaces whose lattice of open sets is a continuous lattice. A continuous lattice is one in which every point is the limit of those way below it. EXPANDALLTHIS

Proposition 5. *The open filter space $\boldsymbol{\phi}X$ is a retract of the Sierpinski hull $\mathbb{S}X$ for every topological space $\boldsymbol{X}$.*

Proof. TBP □

6.1 the injective extension ϕX

- The strict extension whose trace system is all open filters is symbolized $\boldsymbol{\phi}X$ and is sometimes described as "the ultimate monad" in categorical topology.
- It is an injective space and defines a functor from Top0 to the category of injective spaces and maps.
- It is not an idempotent extension, except in the case of a well-ordered poset.
- Every **OF** is a **UF** if and only if the space is a well-ordered poset.
- Every **OF** is a **NF** if and only if the space is a well-ordered poset with top.
- Every **OF** has a largest limit if and only if the space is injective.

6.2 continuous lattices

The original discovery ([`?RefWorks:2354`]) by Dana Scott was that a space is injective if and only if its lattice $\boldsymbol{O}(X)$ is a continuous lattice when given the Scott topology, which in our case is the topology $\boldsymbol{O}^{\vee}(X)$, so we will establish this in our frame of reference.

Our plan will be first to show how the inclusion order of open sets becomes the natural order to use on the space of open sets, and how the Scott topology is a natural outgrowth from that.

We note first that if a space is injective then every **OF** has a largest limit so in particular each nonempty open set $\boldsymbol{V}$ gives an open filter $< V >$, and we will write this limit as $\bar{p}(V)$, which defines a function from $\boldsymbol{O}(X)$ to X. The inclusion order $\boldsymbol{U} \leq V$ translates into the reversed order of inclusion between filters $\boldsymbol{<} U >$ and $\boldsymbol{V}$, and taking the largest limit reverses the order again, so it becomes $\bar{p}^U \leq \bar{p}^V$. Thus for an injective space $\boldsymbol{X}$ we now have the function $\bar{p}$ continous from $i\boldsymbol{O}^{\vee}(X)$ to $X^{\vee}$ and we already know then that it is continous in the firm topologies. We now observe that the function which takes a point into its neighborhood filter preserves order and so is continuous in the firm topologies and thus $\bar{p}$ is actually a retraction map.

The way-below relation can be characterized in many ways. We give three that fit in with our topological approach.

Proposition 6. *The following are equivalent for* $\boldsymbol{U}, V \in O(X)$*:*

- $\boldsymbol{U} << V$
- *there is a point* $\boldsymbol{z} \in X$ *for which* $\boldsymbol{U} \leq\uparrow z \leq V$
- *the filter* $< \boldsymbol{U} >$ *has a limit in* $\boldsymbol{V}$
- *every* **OF** *containing* $\boldsymbol{U}$ *has a limit in* $\boldsymbol{V}$

We are now going to show more, we will show that when $\boldsymbol{\Phi}$ is a closed map then the images of the pointed open sets $\boldsymbol{U}^*$, which are a union base for $\boldsymbol{O}(X)$, actually will be pointed open sets, and a union base for $\boldsymbol{X}$.

Proof. STRAIGHTFORWARD,SEE SCRATCH □

So this adds the further information that a space is algebricaly injective if and only it is a closed retract of an algebrically injective space, noting that in this case $\boldsymbol{O}(X)$ is not only continuous it is algebraic.

6.3 injective hull

One question concerning spaces is "when is there an injective hull", defined as an injective extension that is an essential extension. Since it is expected to be essential it must be a subspace of every other extension and thus a subspace of $\boldsymbol{\phi}X$, since it is expected to be injective it must be a retract of $\boldsymbol{\phi}X$.

Thus we have the characterization that a space has an injective hull if and only if its essential extension $\boldsymbol{\epsilon}X$ is a retract of $\boldsymbol{\phi}X$. According to our characterization this is if and only if every **OF** on $\boldsymbol{\epsilon}X$ is a **NF** In particular each nonempty open set $\boldsymbol{V}^* \in \epsilon X$ must generate a **NF**.

- Ershov [?RefWorks:1225] gives the characterization: for each open $\boldsymbol{V}$ and $\boldsymbol{x} \in V$ we have a finite set $\{x_i \leq x\}$ and open sets $\{V_i : x_i \in V_i\}$ with $\cap V_i \subseteq V$.

  This is clearly equivalent to the condition just stated on extensions.

- In [?RefWorks:1247] is a very categorical treatment of injective hulls.

- In [?RefWorks:1237] there is a detailed discussion of injective hulls for continuous posets, along with many related matters.

7 spectral spaces and the prime open filter space

Definition 4. A space is called *spectral* if:

- it is compact;

- it is sober;

- every open set is a union of compact open sets;

- the meet of compact open sets is compact (and of course open).

Hochster [?RefWorks:2336] characterizes the spectral spaces as the prime ideal spaces of commutative rings with unity, as well as in a topological manner.

7.1 the strict extension $\boldsymbol{\pi}X$

The strict extension whose trace filters are the prime open filters is a spectral space, symbolized as $\boldsymbol{\pi}X$, and is a functor from the category Top0 to the category of spectral spaces and maps.

- This space has been encountered by many investigators, and is sometimes seen merely as a compactification, HERRLICH-BENTLEY following the lead of the space of maximal closed (minimal prime open in duality) filters. It turns out that has much stronger properties than mere compactness, and is intimately related to the class of spectral spaces.

- This extension process is not idempotent. Moreover is has a rather confusing status, in that our intuition derived from common constructions such as the Stone-Cech compactification, or even the Wallman compactication, can often lead us astray.

  For example there are spaces that are spectral, but not the prime ideal space of any space (not even their own).

- Every **PF** is a **UF** if and only if the space is hereditarily compact (often called **Noetherian**).

- Every **PF** is a **NF** if and only if the space is sober and hereditarily compact.
- Noetherian spaces are interesting on ther own. In [?RefWorks:2357] they are shown to be universal for compact T1 In [?RefWorks:2366] they are shown WHAT and in [?RefWorks:897] are many interesting results. AMPLIFY THIS

7.2 stably compact

Definition 5. A space is called *stably compact* if it is:

- compact
- locally compact
- sober
- the meet of compact subsets is compact

Proposition 7. *A space is stably compact if and only if it is a retract of its prime open filter space.*

There is a substantial literature on topological duality. We are going to consider only the most prominent version in this paper, because of its close relationship to stably compact spaces. The **deGroot dual** of a space has the compact $\uparrow$ −sets, which are closed under join, as base for closed sets. The following result has have elaborate history: we cite only [?RefWorks:2363].

Proposition 8. *The deGroot dual of a stably compact space is a stably compact space, and the second dual of it is the original space.*

This duality os seen clearly when we consider it in the context of thei extension $\boldsymbol{\pi} X$ and the retraction to a stably compact space.

Proposition 9. *The compact $\uparrow -sets$ in $\boldsymbol{\pi} X$ are precisely the sets $\{V^* : V \in O(X)\}$. Thus the dual of $\boldsymbol{\pi} X$ has as closed base the open base of $\boldsymbol{\pi} X$, and conversely. The two spaces just reverse one another.*

Proposition 10. *When $\boldsymbol{X}$ is locally compact, for every compact $\uparrow -setK$ and open $\boldsymbol{V}_\alpha$ we have a compact $\boldsymbol{L}_\alpha$ and open $\boldsymbol{U}_\alpha$ with $K \subseteq U\alpha \subseteq L_\alpha \subseteq V_\alpha$.*

Proposition 11. *The closed sets of the dual oof $\boldsymbol{X}$ are the intersections of the compact $\uparrow -sets$ with $\boldsymbol{X}$, and their closures in $\boldsymbol{\pi} X$ are the intersections of the $\boldsymbol{V}$.*

Proposition 12. *A base for closed sets in the strict extension of the dual of $\boldsymbol{X}$ is the sets $\boldsymbol{V}^*$ again. In other words the prime filter space of the dual of a space is precisely the dual of its prime filter space.*

Proposition 13. *For a stably compact space $\boldsymbol{X}$ with deGroot dual $\boldsymbol{Y}$, $\boldsymbol{\pi}Y$ is the dual of $\boldsymbol{\pi}X$, the copen sets of $\boldsymbol{\pi}Y$ are the complements of the copen sets of $\boldsymbol{\pi}X$ (and conversely).*

The dualization process just exchanges the pair consisting of an **PF** *and its* **cPF** *dual as filters with the same pair with positions reversed as a pair on the dual of $\boldsymbol{X}$.*

Proposition 14. *A prime open filter on a copen set has a largest limit.*

Proposition 15. *An open set is compact if and only if every prime open filter that contains it has a limit in the set.*

7.3 prime open filters

Proposition 16. *A prime open filter in a spectral space has a largest limit*

Proof. A spectral space is compact, so a prime open filter has at least one limit. Let $\boldsymbol{A}$ be its set of limit points, and consider the collection of all compact open sets that have non-empty intersection with that. The filter has a limit in each of these, and thus every compact open set that meets $\boldsymbol{A}$ contains a limit point of the prime open filter, so the collection is a union filter. Thus it is a closed-irreducible set, and since spectral spaces are sober it is a generic set. □

Remark 5. Notice that the filter is in fact the copen kernel of the prime open filter.

8 Essentially Complete Spaces

Definition 6. A space is called *essentially complete* if every join filter is a neighbhorhood filter ($\boldsymbol{J}F \subseteq NF$).

Proposition 17. *A space is* essentially complete *if and only if every essential set is generic.*

The strict extension whose trace system is the essential filters is symbolized as $\boldsymbol{\epsilon}X$. It was introduced in [`?RefWorks:2358`]. It is essentially complete, and defines a functor from Top0 to EComp.

- In [`?RefWorks:2343`] these are described categorically as **Right Kan Spaces** following an approach begun by Martin Escardo REFciteRefWorks:1226. These were also studied as lattices by Rudolf Hoffmann REFciteRefWorks:1237.

9 special classes related to filters

There are 14 inclusion relations of filter classes that do not always occur, and their study gives us insight into the meanings of more common type of completeness. We will consider them in order more or less, from the strongest to the weakest.

- **OF** $\subseteq$ **UF** The space is **a well-ordered poset**.
- **OF** $\subseteq$ **NF** The space is **a well-ordered poset bounded above**.
- **OF** $\subseteq$ **PF** $\cap$ **JF**
- **OF** $\subseteq$ **PF the open sets are a chain in inclusion order**
- **OF** $\subseteq$ **JF** It turns out here that every nonempty open set is a meet of saturations, and every **OF** is a union of these.

  Proof. A nonempty $\boldsymbol{V}$ generates a **JF** and so must be of the form $\cap V_i$ where each $\boldsymbol{V}_i$ contains a limit $\boldsymbol{x}_i$ of the filter, and that forces $\boldsymbol{V} = \cap \uparrow x_i = \cap V_i$. But now if $\cap \uparrow x_i \subseteq P$ for some open set we have $\boldsymbol{P} = \cap V_i \cup P$ and so is in the filter. $\square$

- **PF** $\subseteq$ **UF** The space is **hereditarily compact** (often called Noetherian).
- **PF** $\subseteq$ **NF** The space is **sober and hereditarily compact**.
- **PF** $\subseteq$ **JF**
- **JF** $\subseteq$ **UF**
- **JF** $\subseteq$ **PF**
- **PF** $\cap$ **JF** $\subseteq$ **NF**
- **PF** $\cap$ **JF** $\subseteq$ **UF**

9.1 other significant relations

- **CF** $\subseteq$ **OF** always.
- **OF** $\subseteq$ **CF** The space **has a bottom**
- **JF** $\subseteq$ **CF**
- **UF** $\subseteq$ **JF** $\cap$ **PF** $\subseteq$ **CF**
- A point is the bottom iff it has only one nbd.

- Every **JF** is the union of **JF** generated by a finite set of points (**fJF**s.
- There is a largest **JF** enclosed by any **CF**.
- The union and the intersection of **PF** are **PF**.
- Thus there are **PF** maximal and minimal in various situations.
- There is a $\mathbf{JF} \cap \mathbf{PF}$ that is not a **UF**.
- $\mathbf{JF} \subseteq \mathbf{UF}$ implies meet of copen is copen.
- If the space is locally copen then $\mathbf{PF} \subseteq \mathbf{UF}$.
- Every **NF** is the copen kernel of a **PF** iff the space is spectral.
- Every **NF** is the compact kernel of a **PF** iff the space is stably compact.

10 core-compact spaces

When we begin to consider topologies on the collection of open subsets of a space there are many possibilites We will focus only on specific settings that yield a characterization of some class.

Lets begin with the set $\boldsymbol{O}(\pi X)$. Given $\boldsymbol{V} \in O(X)$ define $\boldsymbol{V}^{>} = \{W^+ : V^* \subset W^+\}$.

Noting $\boldsymbol{U}^{>} \cap V^{>} = (U \cap V)^{>}$ we see that we can use these sets as a base for a topology on the set $\boldsymbol{O}(X)$.

These sets are pointed and open, and thus (very) copen. The copen sets are the joins of such sets (and could be described as the *finite bottomed* open sets and the union and intersection of finite collections are also in the collection. The entire space is $\boldsymbol{X}^{>}$ and thus compact.

Given a union filter it also has a base of these pointed sets, and is the neighborhood filter of the open set $\cup\{V^\alpha\}$ for these points, so the space is sober.

We write $\boldsymbol{O}^{>}(\pi X)$ for this ground set with this topology. and note that we have shown the space to be spectral. We note the existence of a function $\boldsymbol{\Delta}$ (not necessarily continuous) which we call *the restriction function* that takes each open set $\boldsymbol{V}^+$ of $\boldsymbol{O}(\pi X)$ to the open subset $\boldsymbol{V}$ of X that it extends.

We shall show that there is a topology on the set $\boldsymbol{O}(X)$ for which this function is continuous with domain $\boldsymbol{O}^{>}(\pi X)$ if and only if every V has a smallest extension ($\boldsymbol{V}^*$ is of course its largest extension) which we call $\boldsymbol{V}^0$, and that this is also equivalent to the core-compactness of $\boldsymbol{X}$, which is in turn equivalent to

In this case the function $\boldsymbol{\Delta}$ will be a retraction and the function $\boldsymbol{\Gamma}$ which takes $\boldsymbol{V} \in O(X)$ to $\boldsymbol{V}^0 \in O^{>}(\pi X)$ is continuous and the section to the retraction.

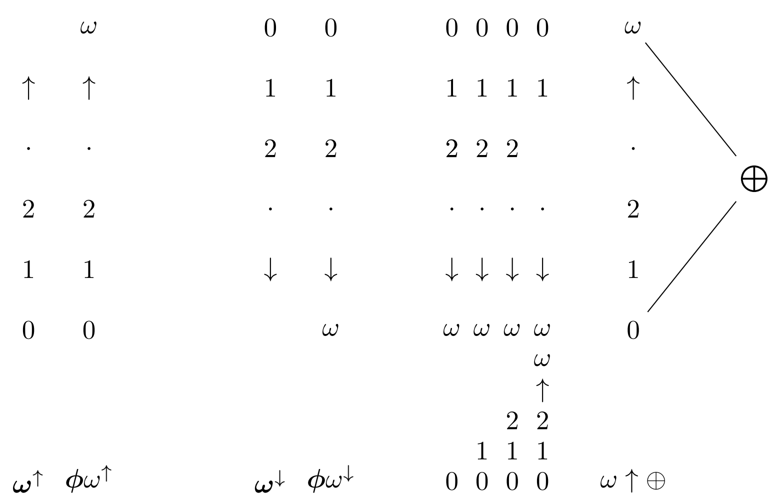
The quotient topology induced on $\boldsymbol{O}(X)$ will have the very interesting property, a generalization of local compactness, that for any $\boldsymbol{V} \in O(x)$ there is $\boldsymbol{U} \in O(x)$ such that $\boldsymbol{U}^* \subset V^0$. Equivalently every open cover of $\boldsymbol{V}$ has a finite subset that covers $\boldsymbol{U}$.

11 EXAMPLES

We begin with an interesting set of five spaces, shown in [?RefWorks:2335] to be the smallest set of spaces such that every infinite space contains one as a subspace. We can view them all as topologies on the countable set $\boldsymbol{\omega}$:

- discrete, which we will view below as $\boldsymbol{\omega}^{\vee}$;
- indiscrete: which we rule out due to our restriction to $\boldsymbol{T}_0$;
- finite complement, which we see as $\boldsymbol{\omega}^{\wedge}$;
- initial segment, which for us is $\boldsymbol{\omega}^{\downarrow}$;
- initial segment, which for us is $\boldsymbol{\omega}^{\uparrow}$.

- Lay the integers out as just a countable set, and we have a specialization class running from $\mathbf{N}^{\vee}$, the **countable discrete space**, to $\mathbf{N}^{\wedge}$, the **countable finite complement space**. The various completions generate a wide varietyi of other spaces.



11.1 $\boldsymbol{\omega}^{\uparrow}$

- This is just the ascending integers and only has one compatible topology.
- It is hereditarily compact, but not sober, since the entire space is irreducible but not generic.

- The collection $\boldsymbol{O}^+(X)$ of nonempty open sets is a **PF** that is not a **NF**, although if is a **UF**. It is also a **JF** so the space is not **essentially complete**.

- The strict completion in every case is done by adding a top point as a non-isolated point, and can be written as $\phi\omega^\uparrow$.

- It is worth noting that there is another point-compatible topology on the completion which makes the top point open, but of course the remainder of the space is closed so it is not a dense extension, thus not a strict extension. This illuminates the fact that we have mentioned that none of our extensions is a $\boldsymbol{T}_d$ space.

11.2 $\omega^\downarrow$

- The space is just the descending integers and again has only one compatible topology. It is in some sense more interesting than its dual, just discussed, because it is the prototypical example of a space whose completions never stop, and in fact in this case generate the set of ordinal numbers as its completions.

- There is only one **OF**, namely $< \omega >$ which is not a **NF**, and it is a **PF** but not a **JF** or a **UF**. It is very much a **SF**, but it is one with no limit, since there is no bottom of the space.

- The space is sober and essentially complete. When we construct either of the strict completions ϕX or πX (they are the same) we add a bottom point and $< \omega >$ is still the only filter of interest. Doing this again just adds a new point above the old bottom and above everything else, and when we have done this ω times we have what is usually called the ordinal w_0 (the ascending integers), Continuing just adds a top to that, which is $\omega+1$. This familiar process of constructing the ordinal numbers and never stops.

- Note of course that all of these retract to the original completion in the obvious way.

11.3 NupPlus

- Adding a point which is above the bottom 0 and below the top ω gives us our first non-familiar example. Then the collection of all intervals $\{n, \omega : n \in \omega\}$ is an **OF** but not a **NF**, so the space is not injective, and every **JF** is a **NF**, so the space is essentially complete.

References